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Some equivalence theorems with K-functionals ☆

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Abstract

The generalized Riesz means, which were introduced by Ditzian (Acta Math. Hungar. 75 (1997) 165), are shown to be equivalent to the corresponding K-functionals in a general setting. Similar results are also obtained for the Cesàro means.

Keywords: Generalized Riesz means; Cesàro means; K-functionals; Strong converse inequality

1. General notations and assumptions

We first introduce some notations from [3]. Let S be a nonempty set equipped with a positive measure μ and let $L^p(S)$, $(1 \leqslant p \leqslant \infty)$ denote the space of functions on S with the usual norm $\|f\|_p = (\int_S |f|^p \, d\mu(x))^{\frac{1}{p}}$, $1 \leqslant p < \infty$ and $\|f\|_\infty := \operatorname{ess\,sup}_{x \in S} |f(x)|$.

Suppose $P(D)$ is a self-adjoint, unbounded operator on $L^2(S)$. We make the following assumptions on $P(D)$:

(i) $P(D)$ has only discrete spectrum $\{-\lambda(k)\}_{k=0}^{\infty}$ and each eigenvalue $-\lambda(k)$ corresponds to a finite-dimensional eigenspace H_k.

(ii) $0 = \lambda(0) < \lambda(1) < \cdots < \lambda(k) < \cdots$ and $\lambda(k)$ is a polynomial in k.

(iii) For some fixed $p \in [1, \infty]$, $H_k \subset L^p(S) \bigcap L^{p'}(S)$ and

$$\overline{\operatorname{span} \bigcup_k H_k} = L^p(S).$$

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Throughout the rest of this paper, the letter B always denotes the space $L^p(S)$ with $p\in[1,\infty]$ satisfying assumption (iii) above and with $\|\cdot\|$ denoting the norm $\|\cdot\|_p$.

Under the above assumptions, the projection P_kf of $f\in B$ on H_k is obviously well defined. For the formal expansion

$$f\sim\sum_{k=0}^{\infty}P_kf,$$

we define its ℓth order Cesàro means (as usual) by

$$\sigma_N^\ell(f)=\sum_{k=0}^{N}\frac{A_{N-k}^\ell}{A_N^\ell}P_kf,$$

where

$$A_k^\ell=\frac{\Gamma(k+\ell+1)}{\Gamma(k+1)\Gamma(\ell+1)}.$$

We make the following additional assumption on the Cesàro means:

(iv) For some $\ell=\ell(B)\in\mathbb{N}$,

$$\sup_N\|\sigma_N^\ell(f)\|\leqslant C(\ell,B)\|f\|. \tag{1.1}$$

We remark that as pointed out in [1,3], the above assumptions are very natural and many differential operators and the expansions related to them, such as spherical harmonics and the Laplace–Beltrami operator, Jacobi expansions and the Jacobi operator, and Hermite and Laguerre expansions and their operators, satisfy these assumptions.

Now let us define the fractional differential operator $P(D)^\alpha$ (for a given α), in the sense of distributions, by

$$P(D)^\alpha f\sim\sum_{k=0}^{\infty}(-\lambda(k))^\alpha P_kf.$$

We write $(-P(D))^\alpha f=f^{(\alpha)}$ if $P(D)^\alpha f\in B$. To each operator $(-P(D))^\alpha$ is associated with a K-functional

$$K_\alpha(f,t)=\inf\{\|f-g\|+t\|g^{(\alpha)}\|:\ g^{(\alpha)}\in B\}.$$

It will be convenient to use the notation

$$A(f,t)\approx B(f,t),$$

which means that there is a $C>0$, independent of f and t, such that

$$C^{-1}A(f,t)\leqslant B(f,t)\leqslant CA(f,t).$$

A strong converse inequality of type B (in the sense of [4]) is a result of the type

$$\|T_t(f)-f\|+\|T_{\frac{t}{M}}(f)-f\|\approx K_r(f,t) \tag{1.2}$$

and a strong converse inequality of type A will be (1.2) when we can drop the second term on its left-hand side. Here $\{T_t\}_{t>0}$ is a given family of continuous linear operators on B.

For further details of the background information, we refer the reader to [2,5–7] and to [1,3], where many impressive results were obtained in the above setting and many interesting applications were given to some known differential operators and the expansions related to them.

Except when otherwise stated, the letter C denotes a general constant depending only on the parameters indicated as subscripts, and possibly also on the space B and the operator $P(D)$.

2. Riesz means

For $\lambda > 0$, $\alpha > 0$ and $\ell \in \mathbb{N}$, the generalized Riesz means, which were introduced in [3], are defined by

$$R_{\lambda,\alpha,\ell}(f) = \sum_{\lambda(k)<\lambda} \left(1 - \left(\frac{\lambda(k)}{\lambda}\right)^{\alpha}\right)^{\ell} P_k(f).$$

It follows from [3] that under assumption (iv),

$$\sup_{\lambda>0} ||R_{\lambda,\alpha,\ell}(f)|| \leqslant C||f|| \tag{2.1}$$

with C independent of f.

In this section, we shall prove the following theorem, which was conjectured in [3] under hypothesis (2.1):

Theorem 2.1. *Suppose $\ell \in \mathbb{N}$ and* (1.1) *is satisfied. Then for $\lambda > 0$, $\alpha > 0$ and $m \in \mathbb{N}$,*

$$||(R_{\lambda,\alpha,\ell} - I)^m f|| \approx K_{\alpha m}(f, \lambda^{-\alpha m}).$$

Theorem 2.1 for $\ell = 1$ is due to [3]. For $\ell \geqslant 2$, a result of type B like (1.2) was obtained in [1, (3.10), p. 181]; [3, (5.7), p. 335]. For all $\ell \geqslant 1$, it was shown in [3, (5.1), p. 334] that

$$||(R_{\lambda,\alpha,\ell} - I)^m f|| + \lambda^{-\alpha m}||(R_{\lambda,\alpha,\ell,m} f)^{(\alpha m)}|| \approx K_{\alpha m}(f, \lambda^{-\alpha m}), \tag{2.2}$$

where the operator $R_{\lambda,\alpha,\ell,m}$ is defined by

$$R_{\lambda,\alpha,\ell,m} = I - (I - R_{\lambda,\alpha,\ell})^m = \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} R^k_{\lambda,\alpha,\ell}. \tag{2.3}$$

To prove Theorem 2.1, we need the following lemmas.

Lemma 2.2. *Suppose $\ell \in \mathbb{N}$ is as in assumption* (iv) *and $\eta \in C^{(\ell+1)}(\mathbb{R}_+)$ is of compact support. For $\lambda > 0$, define*

$$V_\lambda(f) = \sum_{k=0}^{\infty} \eta\left(\frac{\lambda(k)}{\lambda}\right) P_k(f).$$

Then

$$||V_\lambda(f)|| \leqslant C_\eta ||f||,$$

with $C_\eta > 0$ *independent of* f *and* $\lambda > 0$.

Proof. Suppose $\operatorname{supp} \eta \subset [0, a]$ with $a > 0$ depending only on η, and suppose $\lambda(n_0 - 1) \leqslant a\lambda < \lambda(n_0)$ with $n_0 \in \mathbb{N}$. Noticing that

$$||P_k f|| = \left|\left| \overleftarrow{\triangle}^{\ell+1} \binom{k+\ell}{\ell} \sigma_k^\ell(f) \right|\right| \leqslant C(k+1)^\ell ||f||,$$

by assumption (ii), without loss of generality, we may assume the function $\lambda(x)$ is strictly increasing on $[0, \infty)$.

By (1.1) and the Abel transformation, it suffices to prove

$$\sum_{k=0}^{n_0} \left| \triangle^{\ell+1} \eta\left(\frac{\lambda(k)}{\lambda}\right) \right| k^\ell \leqslant C_\eta. \tag{2.4}$$

Let $\varphi(x) = \eta(\frac{\lambda(x)}{\lambda})$. Then a straightforward computation shows that

$$|\varphi^{(\ell+1)}(x)| \leqslant C_\eta \sum_{i=1}^{\ell+1} \left(\frac{\lambda(x)}{\lambda}\right)^i \frac{1}{(x+1)^{\ell+1}}. \tag{2.5}$$

Noticing that

$$\triangle^{\ell+1} \eta\left(\frac{\lambda(k)}{\lambda}\right) = \varphi^{(\ell+1)}(\theta_k)$$

for some $\theta_k \in [k, k+\ell+1]$, we get from (2.5)

$$\left| \triangle^{\ell+1} \eta\left(\frac{\lambda(x)}{\lambda}\right) \right| \leqslant C_\eta \sum_{i=1}^{\ell+1} \left(\frac{\lambda(\theta_k)}{\lambda}\right)^i \frac{1}{\theta_k^{\ell+1}}. \tag{2.6}$$

Now substituting (2.6) into the left-hand side of (2.4), taking into account the monotonicity of $\lambda(x)$, we obtain (2.4) and complete the proof. $\square$

Lemma 2.3. *Suppose* $\lambda > 0$ *and* $R_{\lambda,\alpha,\ell,m} f$ *is defined by* (2.3). *Then*

$$\left|\left| \left(\frac{1}{\lambda}\right)^{\alpha m} (R_{\lambda,\alpha,\ell,m} f)^{(\alpha m)} \right|\right| \leqslant C ||(I - R_{\lambda,\alpha,\ell})^m f||,$$

with $C > 0$ *independent of* λ *and* f.

Proof. By (2.3) and (2.1), it is sufficient to prove

$$\left|\left| \left(\frac{1}{\lambda}\right)^{\alpha m} (R_{\lambda,\alpha,\ell} f)^{(\alpha m)} \right|\right| \leqslant C ||(I - R_{\lambda,\alpha,\ell})^m f||. \tag{2.7}$$

We begin by fixing η, a C^∞ function of compact support, defined on $\mathbb{R}$, with the properties that $\eta(x) = 1$ for $|x| \leqslant \frac{1}{4}$ and $\eta(x) = 0$ for $|x| \geqslant \frac{1}{2}$. Let

$$a(k, \lambda) = \left(1 - \left(\frac{\lambda(k)}{\lambda}\right)^\alpha\right)^\ell.$$

We decompose the operator $(\frac{1}{\lambda})^{\alpha m}(R_{\lambda,\alpha,\ell}f)^{(\alpha m)}$ as

$$\left(\frac{1}{\lambda}\right)^{\alpha m} (R_{\lambda,\alpha,\ell}f)^{(\alpha m)} = T_\lambda^1 f + T_\lambda^2 f, \tag{2.8}$$

where

$$T_\lambda^1 f = \sum_{\lambda(k)<\lambda} a(k, \lambda)\left(\frac{\lambda(k)}{\lambda}\right)^{\alpha m} \eta\left(\frac{\lambda(k)}{\lambda}\right) P_k f,$$

$$T_\lambda^2 f = \sum_{\lambda(k)<\lambda} a(k, \lambda)\left(\frac{\lambda(k)}{\lambda}\right)^{\alpha m} \left(1 - \eta\left(\frac{\lambda(k)}{\lambda}\right)\right) P_k f.$$

First, we will prove for $i = 1, 2$

$$||T_\lambda^i f|| \leqslant C||(I - R_{\lambda,\alpha,\ell})^m f||, \tag{2.9}$$

with $C>0$ independent of λ and f. For $i = 1$, let us rewrite $T_\lambda^1 f$ as

$$T_\lambda^1 f = \sum_{\lambda(k)<\lambda} a(k, \lambda)\xi\left(\frac{\lambda(k)}{\lambda}\right) P_k(h), \tag{2.10}$$

where

$$\xi(t) = \frac{\eta(t)t^{\alpha m}}{(1 - (1 - t^\alpha)^\ell)^m}, \quad \text{and} \quad h = (I - R_{\lambda,\alpha,\ell})^m f. \tag{2.11}$$

Noticing that

$$\xi(t) = \frac{\eta(t)}{\left(\ell + \sum_{j=1}^{\ell-1} (-1)^j \begin{pmatrix} \ell \\ j+1 \end{pmatrix} t^{\alpha j}\right)^m} \in C_0^\infty(\mathbb{R}_+)$$

with supp $\xi \subset \{t\colon 0 \leqslant t \leqslant \frac{1}{2}\}$, we obtain (2.9) for $i = 1$, by Lemma 2.2, (2.1) and (2.10)–(2.11).

Next, we prove (2.9) for $i = 2$. Define

$$U_\lambda(g) = \sum_{\lambda(k)<\lambda} \left(1 - \eta\left(\frac{\lambda(k)}{\lambda}\right)\right) \frac{a(k, \lambda)}{(1 - a(k, \lambda))^m} P_k(g).$$

Below we will prove

$$||U_\lambda(g)|| \leqslant C||g||, \tag{2.12}$$

with $C>0$ independent of λ and g.

We rewrite $U_\lambda(g)$ as

$$U_\lambda(g) = U^1_\lambda(g) + U^2_\lambda(g),$$

where

$$U^1_\lambda(g) = \sum_{\lambda(k)<\lambda} \left(1-\eta\left(\frac{\lambda(k)}{\lambda}\right)\right) a(k,\lambda)[1+ma(k,\lambda)]P_k(g),$$

$$U^2_\lambda(g) = \sum_{\lambda(k)<\lambda} \left(1-\eta\left(\frac{\lambda(k)}{\lambda}\right)\right) a(k,\lambda)\left[\frac{1}{(1-a(k,\lambda))^m} - 1 - ma(k,\lambda)\right]P_k(g).$$

By Lemma 2.2 and (2.1), one can easily verify for $i=1$

$$||U^i_\lambda(g)|| \leqslant C||g||. \tag{2.13}$$

To deal with U^2_λ, we set

$$\varphi(t) = \begin{cases} (1-\eta(t))(1-t^\alpha)^\ell \left[\frac{1}{(1-(1-t^\alpha)^\ell)^m} - 1 - m(1-t^\alpha)^\ell\right], & \text{if } 0\leqslant t\leqslant 1,\\ 0 & \text{if } t\geqslant 1. \end{cases}$$

Then, a straightforward computation shows

$$\varphi(t) = (1-\eta(t))\frac{(1-t^\alpha)^{3\ell}\sum_{j=0}^{(m-1)\ell} C_{m,\ell,j}t^{\alpha j}}{(1-(1-t^\alpha)^\ell)^m}, \quad \frac{1}{4}\leqslant t\leqslant 1,$$

where the $C_{m,\ell,j}$ are constants depending only on m, ℓ and j. This clearly implies $\varphi \in C_0^{(\ell+1)}(\mathbb{R}_+)$. Noticing that

$$U^2_\lambda(g) = \sum_{k=0}^{\infty} \varphi\left(\frac{\lambda(k)}{\lambda}\right)P_k(g),$$

by Lemma 2.2, we get (2.13) for $i=2$. Putting this together, we get (2.12).

Now noticing that

$$T^2_\lambda(f) = \left(\frac{1}{\lambda}\right)^{\alpha m}(U_\lambda(I-R_{\lambda,\alpha,\ell})^m f)^{(\alpha m)},$$

by Bernstein's inequality (see [3, (3.5), p. 330]), we get from (2.1) and (2.12)

$$||T^2_\lambda(f)|| \leqslant C||(I-R_{\lambda,\alpha,\ell})^m f||,$$

which, together with (2.9) and (2.8), yields (2.7). This completes the proof. □

Now Theorem 2.1 is an immediate consequence of Lemma 2.3 and (2.2).

3. Cesàro means

In this section, we prove

Theorem 3.1. *Suppose* $\ell \in \mathbb{N}$ *and* (1.1) *is satisfied. Then*

$$\|f - \sigma_N^\ell(f)\| \approx K_{\alpha_0}\left(f, \frac{1}{N}\right),$$

where $\alpha_0 = (\deg \lambda(x))^{-1}$ *and* $\lambda(x)$ *is as in assumption* (ii).

Throughout the rest of this section, the symbol α_0 will always denote the number $\frac{1}{\deg \lambda(x)}$. The proof of Theorem 3.1 is based on the following lemmas.

Lemma 3.2. *Suppose* $\varphi(x), \phi(x)$ *are two algebraic polynomials of the same degree. Assume there exists a positive integer* n_0, *such that* $\phi(x)\varphi(x) > 0$ *whenever* $x \geqslant n_0$. *For a given* $r > 0$, *define*

$$T(f) = \sum_{k=n_0}^{\infty} \left(\frac{\varphi(k)}{\phi(k)}\right)^r P_k f.$$

Then

$$\|T(f)\| \leqslant C(\varphi, \phi, r, n_0)\|f\|.$$

Proof. Let

$$\Psi(x) = \left(\frac{\varphi(x)}{\phi(x)}\right)^r, \quad x \geqslant n_0.$$

Noticing that $\varphi(x)$ and $\phi(x)$ are polynomials of the same degree, one can easily verify that

$$|\Psi^{(\ell+1)}(x)| \leqslant C\left(\frac{1}{x+1}\right)^{\ell+2}, \quad x \geqslant n_0. \tag{3.1}$$

Now let us define

$$\mu_k = \begin{cases} \Psi(k), & k \geqslant n_0, \\ 0, & 0 \leqslant k < n_0. \end{cases}$$

Using Abel's transformation $\ell + 1$ times, taking into account (1.1), we obtain

$$\|T(f)\| \leqslant C_B \sum_{k=0}^{\infty} |\triangle^{\ell+1}\mu_k| k^\ell \|f\|,$$

which, by (3.1), implies the desired result. $\square$

Lemma 3.3.

$$\|f - \sigma_N^\ell(f)\| \leqslant CK_{\alpha_0}\left(f, \frac{1}{N}\right),$$

with $C > 0$ independent of N and f.

Proof. We get the idea from [3]. Let $g = R_{\lambda(\frac{N}{2}),\alpha_0,\ell,1} f$ with $R_{\lambda(\frac{N}{2}),\alpha_0,\ell,1}$ defined as in (2.3). Observing that $g \in \oplus_{k=0}^{\frac{N}{2}} H_k$ and $(\lambda(\frac{N}{2}))^{\alpha_0} \sim N$, we get from (2.2) that

$$\|f - g\| + \frac{1}{N}\|g^{(\alpha_0)}\| \leqslant CK_{\alpha_0}\left(f, \frac{1}{N}\right).$$

On the other hand, by (1.1),

$$\begin{aligned}\|\sigma_N^\ell(f) - f\| &\leqslant \|\sigma_N^\ell(f) - \sigma_N^\ell(g)\| + \|\sigma_N^\ell(g) - g\| + \|g - f\| \\ &\leqslant C\|f - g\| + \|\sigma_N^\ell(g) - g\|.\end{aligned}$$

Hence, it suffices to prove that

$$\|\sigma_N^\ell(g) - g\| \leqslant C\frac{1}{N}\|g^{(\alpha_0)}\|. \tag{3.2}$$

From assumptions (iii) and (iv), it follows that

$$\lim_{N\to\infty} \|\sigma_N^\ell(g) - g\| = 0,$$

which implies

$$\begin{aligned}\sigma_N^\ell(g) - g &= \sum_{k=N}^{\infty} (\sigma_k^\ell(g) - \sigma_{k+1}^\ell(g)) \\ &= -\sum_{k=N}^{\infty} \frac{1}{(k+1+\ell)(k+1)} \sum_{j=0}^{\frac{N}{2}} \frac{A_{k-j}^\ell}{A_k^\ell} \frac{j\ell(k+1)}{k-j+1} P_j(g).\end{aligned} \tag{3.3}$$

Let $\eta \in C_0^\infty(\mathbb{R}_+)$ be a C^∞ function, defined on R, with the properties that $\eta(x) = 1$ for $|x| \leqslant \frac{1}{2}$ and $\eta(x) = 0$ for $|x| \geqslant \frac{3}{4}$. Then, noticing that $g \in \oplus_{k=0}^{\frac{N}{2}} H_k$, by (1.1) and

Lemma 3.2, we have

$$\left\|\sum_{j=0}^{\frac{N}{2}} \frac{A_{k-j}^{\ell}}{A_k^{\ell}} \frac{j\ell(k+1)}{k-j+1} P_j(g)\right\| = \left\|\sum_{j=0}^{\frac{N}{2}} \frac{A_{k-j}^{\ell}}{A_k^{\ell}} \frac{j\ell(k+1)}{k-j+1} \eta\left(\frac{j}{N}\right) P_j(g)\right\|$$
$$\leqslant C\left\|\sum_{j=0}^{\frac{N}{2}} \frac{k+1}{k-j+1} \eta\left(\frac{j}{N}\right) P_j(g^{(\alpha_0)})\right\|$$
$$\leqslant C\sum_{j=0}^{\frac{3}{4}N} \left|\vec{\triangle}^{\ell+1}\left(\eta\left(\frac{j}{N}\right)\frac{k+1}{k+1-j}\right)\right| \times (j+1)^{\ell}\|g^{(\alpha_0)}\|. \tag{3.4}$$

A straightforward computation shows that

$$\left|\vec{\triangle}^{\ell+1}\left(\eta\left(\frac{j}{N}\right)\frac{k+1}{k+1-j}\right)\right| \leqslant C\left(\frac{1}{N}\right)^{\ell+1}, \quad 0 \leqslant j \leqslant \frac{3}{4}N \leqslant \frac{3}{4}k. \tag{3.5}$$

Now substituting (3.5) into (3.4), taking into account (3.3), we obtain (3.2) and complete the proof. □

Lemma 3.4.

$$\left(\frac{1}{\lambda(N)}\right)^{\alpha_0} \|(\sigma_N^{\ell}(f))^{(\alpha_0)}\| \leqslant C\|f - \sigma_N^{\ell}(f)\|,$$

with C independent of N and f.

Now Theorem 3.1 follows directly from Lemmas 3.3 and 3.4 and the fact that $\lambda(N)^{\alpha_0} \sim N$. So, it remains to prove Lemma 3.4. To this end, we need some additional lemmas.

Lemma 3.5. *Let*

$$a_k = \begin{cases} \frac{N}{k}\left(1 - \frac{A_{N-k}^{\ell}}{A_N^{\ell}}\right), & 1 \leqslant k \leqslant N, \\ 0, & k \geqslant N+1. \end{cases}$$

Then for $i = 0, 1, \ldots, \ell+1$ *and* $1 \leqslant k \leqslant [\frac{3}{4}N] + 1$,

$$|\vec{\triangle}^{i} a_k| \leqslant C\left(\left(\frac{1}{N}\right)^{i} + \left(\frac{1}{k+1}\right)^{i+1}\right).$$

Proof. We rewrite a_k as

$$a_k = b_k + c_k, \tag{3.6}$$

where

$$b_k = \frac{N}{k}\left(1 - \left(1 - \frac{k}{N}\right)^{\ell}\right),$$
$$c_k = \frac{N}{k}\left(\left(1 - \frac{k}{N}\right)^{\ell} - \frac{A^{\ell}_{N-k}}{A^{\ell}_N}\right). \tag{3.7}$$

Let $\varphi(t) = \frac{1}{t}(1-(1-t)^{\ell})$. Noticing that

$$\varphi(t) = \sum_{j=1}^{\ell} \binom{\ell}{j} (-1)^{j-1} t^{j-1} \in C^{\infty}[0, \infty]$$

and $b_k = \varphi(\frac{k}{N})$, we get for $i \in \mathbb{Z}_+$,

$$|\vec{\Delta}^i b_k| \leqslant C\left(\frac{1}{N+1}\right)^i, \quad 1 \leqslant k \leqslant N.$$

Hence, by (3.6), it remains to show for $i = 0, 1, \dots, \ell+1$,

$$|\vec{\Delta}^i c_k| \leqslant C\left(\frac{1}{k+1}\right)^{i+1}, \quad 1 \leqslant k \leqslant \left[\frac{3}{4}N\right] + 1.$$

On account of (3.7), it suffices to prove for $0 \leqslant i \leqslant \ell+1$,

$$\left|\vec{\Delta}^i\left(\left(1 - \frac{k}{N}\right)^{\ell} - \frac{A^{\ell}_{N-k}}{A^{\ell}_N}\right)\right| \leqslant C_i \frac{1}{(N+1)^{i+1}}, \quad 1 \leqslant k \leqslant \left[\frac{3}{4}N\right] + 1. \tag{3.8}$$

Noticing that

$$\vec{\Delta} A^{\delta}_{N-k} = A^{\delta-1}_{N-k}, \quad \delta > -1, \tag{3.9}$$

and (see [8, p. 77, (1.18)])

$$A^{\delta}_k = \frac{k^{\delta}}{\Gamma(\delta+1)}\left(1 + O\left(\frac{1}{k}\right)\right), \quad \delta > -1, \ k \in \mathbb{N},$$

we get for $0 \leqslant k \leqslant [\frac{3}{4}N] + 1$,

$$\vec{\Delta}^i \frac{A^{\ell}_{N-k}}{A^{\ell}_N} = \frac{\vec{\Delta}^i A^{\ell}_{N-k}}{A^{\ell}_N} = \begin{cases} \frac{A^{\ell-i}_{N-k}}{A^{\ell}_N} = \frac{\Gamma(\ell+1)}{\Gamma(\ell+1-i)} \frac{(N-k)^{\ell-i}}{N^{\ell}}(1 + O(\frac{1}{N})) & \text{if } 0 \leqslant i \leqslant \ell, \\ 0 & \text{if } i = \ell+1. \end{cases} \tag{3.10}$$

On the other hand, it is easy to verify that for $0 \leqslant k \leqslant [\frac{3}{4}N] + 1$,

$$\vec{\Delta}^i \left(1 - \frac{k}{N}\right)^{\ell} = \begin{cases} \frac{\Gamma(\ell+1)}{\Gamma(\ell-i+1)}(\frac{1}{N})^i (1 - \frac{k+\theta_{i,k}}{N})^{\ell-i} & \text{if } 0 \leqslant i \leqslant \ell, \\ 0 & \text{if } i = \ell+1, \end{cases} \tag{3.11}$$

with

$$0 < \theta_{i,k} < i, \quad i = 0, 1, \dots, \ell.$$

Now combining (3.10) and (3.11), we get for $0 \leqslant i \leqslant \ell$ and $0 \leqslant k \leqslant [\frac{3}{4}N] + 1$,

$$\vec{\triangle}^{\,i}\left(\left(1-\frac{k}{N}\right)^{\ell}-\frac{A_{N-k}^{\ell}}{A_N^{\ell}}\right) = \frac{\Gamma(\ell+1)}{\Gamma(\ell-i+1)}\frac{1}{N^i}\left(\left(1-\frac{k+\theta_i}{N}\right)^{\ell-i} - \left(1-\frac{k}{N}\right)^{\ell-i} + O\left(\frac{1}{N}\right)\right) = O\left(\left(\frac{1}{N}\right)^{i+1}\right),$$

and for $i = \ell + 1$ and $0 \leqslant k \leqslant [\frac{3}{4}N] + 1$,

$$\vec{\triangle}^{\,\ell+1}\left(\left(1-\frac{k}{N}\right)^{\ell}-\frac{A_{N-k}^{\ell}}{A_N^{\ell}}\right) = 0,$$

which gives (3.8) and completes the proof. □

Lemma 3.6. *Suppose* $a_k \geqslant \delta > 0$, $k = 0, 1, \dots$ *Then*

$$\left|\vec{\triangle}^{\,n}\frac{1}{a_k}\right| \leqslant C(\delta, n)\sup\{|\vec{\triangle}^{\,i_1}a_{k+j_1}\cdots\vec{\triangle}^{\,i_m}a_{k+j_m}|:\ 1 \leqslant i_u,\ j_u \leqslant n,\ 1 \leqslant u \leqslant m \leqslant n,\ i_1 + i_2 + \cdots + i_m = n\}.$$

Lemma 3.6 can be easily obtained by induction on n and using the following two identities:

$$\vec{\triangle}\frac{1}{a_k} = -\frac{\vec{\triangle}a_k}{a_k a_{k+1}},$$

$$\vec{\triangle}^{\,n+1}\frac{1}{a_k} = -\vec{\triangle}^{\,n}\left(\frac{\vec{\triangle}a_k}{a_k a_{k+1}}\right) = -\sum_{j=0}^{n}\binom{n}{j}(\vec{\triangle}^{\,n-j+1}a_{k+j})\left(\sum_{i=0}^{j}\binom{j}{i}\left(\vec{\triangle}^{\,i}\frac{1}{a_k}\right)\vec{\triangle}^{\,j-i}\left(\frac{1}{a_{k+1+i}}\right)\right).$$

Lemma 3.7. *Let*

$$\mu_k = \begin{cases} \frac{A_{N-k}^{\ell}}{A_N^{\ell}}, & 0 \leqslant k \leqslant N, \\ 0, & k \geqslant N+1. \end{cases}$$

Then for $i = 0, 1, \dots, \ell + 1$ *and* $\frac{N}{8} \leqslant k \leqslant N$,

$$\left|\vec{\triangle}^{\,i}\left(\frac{\mu_k^2}{1-\mu_k}\right)\right| \leqslant C\left(\frac{1}{N}\right)^{i}. \tag{3.12}$$

Proof. First, we prove for $i = 0, 1, \ldots, \ell$,

$$\left|\vec{\Delta}^i \frac{1}{1-\mu_k}\right| \leqslant C\left(\frac{1}{N}\right)^i, \quad k \geqslant \frac{N}{8}, \tag{3.13}$$

and for $i = \ell + 1$

$$\left|\vec{\Delta}^{\ell+1} \frac{1}{1-\mu_k}\right| \leqslant \begin{cases} C(\frac{1}{N})^{\ell+1}, & \text{if } \frac{N}{8} \leqslant k \leqslant N - \ell - 2, \\ C(\frac{1}{N})^{\ell}, & \text{if } N - \ell - 1 \leqslant k \leqslant N + \ell + 1. \end{cases} \tag{3.14}$$

Since for $i \geqslant 0$

$$\vec{\Delta}^i a_k = \vec{\Delta}^i a_{N+1+\ell} + \sum_{j=k}^{N+\ell} \vec{\Delta}^{i+1} a_j, \tag{3.15}$$

it is sufficient to prove (3.14). By (3.9), it is easy to verify that for $0 \leqslant i \leqslant \ell$,

$$|\vec{\Delta}^i \mu_k| \leqslant C\left(\frac{1}{N}\right)^i, \quad k \geqslant 0, \tag{3.16}$$

and for $i = \ell + 1$

$$\vec{\Delta}^{\ell+1} \mu_k = \begin{cases} \frac{1}{A_N^\ell} = O(\frac{1}{N^\ell}) & \text{if } k = N, \\ 0 & \text{otherwise.} \end{cases} \tag{3.17}$$

A straightforward computation shows that

$$1 - \mu_k \geqslant 1 - \frac{A^\ell_{\frac{7N}{8}}}{A_N^\ell} \geqslant \frac{1}{16} \tag{3.18}$$

whenever $k \geqslant \frac{N}{8}$.

Now applying Lemma 3.6 with $n = \ell + 1$ and $a_k = 1 - \mu_k$, we get, by (3.16) and (3.18),

$$\left|\vec{\Delta}^{\ell+1} \frac{1}{1-\mu_k}\right| \leqslant C\left(\frac{1}{N}\right)^{\ell+1} + C \max_{1 \leqslant j \leqslant \ell+1} |\vec{\Delta}^{\ell+1} \mu_{k+j}|,$$

which, on account of (3.17), gives (3.14) and hence (3.13).

Next, we prove for $i = 0, \ldots, \ell + 1$,

$$|\vec{\Delta}^i \mu_k^2| \leqslant C(\tfrac{1}{N})^i, \quad k \geqslant \tfrac{N}{8}. \tag{3.19}$$

As

$$\vec{\Delta}^i \mu_k^2 = \sum_{j=0}^{k} \binom{i}{j} \vec{\Delta}^{i-j} \mu_k \vec{\Delta}^j \mu_k,$$

(3.19) follows from (3.16) and (3.17).

Finally, we prove (3.12). By (3.15), it suffices to consider the case $i = \ell + 1$. We use the following identity

$$\left|\vec{\triangle}^{\ell+1}\left(\frac{\mu_k^2}{1-\mu_k}\right)\right| = \left|\frac{\vec{\triangle}^{\ell+1}\mu_k^2}{1-\mu_{k+\ell+1}} + \mu_k^2\,\vec{\triangle}^{\ell+1}\left(\frac{1}{1-\mu_k}\right) + \sum_{i=1}^{\ell}\binom{\ell+1}{i}\vec{\triangle}^{i}\mu_k^2\,\vec{\triangle}^{\ell+1-i}\left(\frac{1}{1-\mu_{k+i}}\right)\right|.$$

We then get from (3.16), (3.17) and (3.19)

$$\left|\vec{\triangle}^{\ell+1}\left(\frac{\mu_k^2}{1-\mu_k}\right)\right| \leqslant C\left(\frac{1}{N}\right)^{\ell+1}.$$

This gives (3.12) and completes the proof. □

Proof of Lemma 3.4. Without loss of generality, we may assume $P_0(f) = 0$. Let $\eta \in C_0^\infty(\mathbb{R}_+)$ such that $\eta(x) = 1$ for $|x| \leqslant \frac{1}{4}$ and $\eta(x) = 0$ for $|x| \geqslant \frac{1}{2}$. We decompose $(\frac{1}{\lambda(N)})^{\alpha_0}(\sigma_N^\ell(f))^{(\alpha_0)}$ as

$$\left(\frac{1}{\lambda(N)}\right)^{\alpha_0}(\sigma_N^\ell(f))^{(\alpha_0)} = T_N^1(f) + T_N^2(f), \tag{3.20}$$

where

$$T_N^1(f) := \sum_{k=0}^{N}\left(\frac{\lambda(k)}{\lambda(N)}\right)^{\alpha_0}\frac{A_{N-k}^\ell}{A_N^\ell}\eta\left(\frac{k}{N}\right)P_k(f),$$

$$T_N^2(f) := \sum_{k=0}^{N}\left(\frac{\lambda(k)}{\lambda(N)}\right)^{\alpha_0}\frac{A_{N-k}^\ell}{A_N^\ell}\left(1-\eta\left(\frac{k}{N}\right)\right)P_k(f).$$

We will prove for $i = 1, 2$,

$$\|T_N^i(f)\| \leqslant C\|f - \sigma_N^\ell(f)\|. \tag{3.21}$$

For $i = 1$, by Berntein's inequality and (1.1), we get

$$\|T_N^1(f)\| \leqslant C\left\|\sum_{k=1}^{N}\frac{k}{N}\eta\left(\frac{k}{N}\right)P_k(f)\right\|. \tag{3.22}$$

Define

$$G_N^1(g) = \sum_{k=1}^{N}\eta\left(\frac{k}{N}\right)\frac{\frac{k}{N}}{1-\frac{A_{N-k}^\ell}{A_N^\ell}}P_k(g). \tag{3.23}$$

Observe that for $1 \leqslant k \leqslant N$,

$$\frac{N}{k}\left(1-\frac{A_{N-k}^\ell}{A_N^\ell}\right) \geqslant \frac{N}{k}\left(1-\left(1-\frac{\ell}{N+\ell}\right)^k\right) \geqslant \frac{1}{2}\frac{1}{\ell+1} > 0.$$

From Lemmas 3.5 and 3.6, it follows that

$$\left| \vec{\triangle}^{\ell+1} \left(\eta\left(\frac{k}{N}\right) \frac{1}{\frac{N}{k}\left(1 - \frac{A^{\ell}_{N-k}}{A^{\ell}_{N}}\right)} \right) \right| \leqslant C \left(\left(\frac{1}{k+1}\right)^{\ell+2} + \left(\frac{1}{N}\right)^{\ell+1} \right),$$

which, by (1.1), implies

$$\|G^1_N(g)\| \leqslant C\|g\|. \tag{3.24}$$

Now combining (3.22)–(3.24), we obtain (3.21) for $i = 1$.

For $i = 2$, we define

$$G^2_N(g) = \sum_{k=0}^{N} \frac{A^{\ell}_{N-k}}{A^{\ell}_{N}} \frac{1}{1 - \frac{A^{\ell}_{N-k}}{A^{\ell}_{N}}} \left(1 - \eta\left(\frac{k}{N}\right)\right) P_k(g).$$

We decompose G^2_N as

$$G^2_N(g) = G^{2,1}_N(g) + G^{2,2}_N(g),$$

where

$$G^{2,1}_N(g) := \sum_{k=0}^{N} \frac{\left(\frac{A^{\ell}_{N-k}}{A^{\ell}_{N}}\right)^2}{1 - \frac{A^{\ell}_{N-k}}{A^{\ell}_{N}}} \left(1 - \eta\left(\frac{k}{N}\right)\right) P_k(g),$$

$$G^{2,2}_N(g) := \sum_{k=0}^{N} \frac{A^{\ell}_{N-k}}{A^{\ell}_{N}} \left(1 - \eta\left(\frac{k}{N}\right)\right) P_k(g).$$

From Lemma 3.7 and Abel's transformation, it follows that

$$\|G^{2,1}_N(g)\| \leqslant C\|g\|.$$

On the other hand, by assumption (iv) and Lemma 2.2, it is easy to verify $\|G^{2,2}_N(g)\| \leqslant C\|g\|$.

Thus

$$\|G^2_N(g)\| \leqslant C\|g\|.$$

Observing

$$T^2_N(f) = \left\| \left(\frac{1}{\lambda(N)}\right)^{\alpha_0} (G^2_N(f - \sigma^{\ell}_N(f)))^{(\alpha_0)} \right\|,$$

by Bernstein's inequality (see [3, (3.5), p. 330]), we derive (3.21) for $i = 2$. This completes the proof. $\square$

Remark 3.1. In assumption (iv) the condition "ℓ is a positive integer" can be removed. Indeed, a modification of the above proofs will show that Theorems 2.1 and 3.1 remain valid with ℓ replaced by any positive number δ for which (1.1) is satisfied.

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